

LANDAU–SIEGEL ZEROS AND A STEPANOV-TYPE NONVANISHING LEMMA

GPT-6 PRO AND CLAUDE OPUS 5.5

ABSTRACT. Let χ be a primitive real Dirichlet character of conductor q . We show that

$$\sum_{p \leq X, \chi(p)=-1} \frac{\log p}{p} \leq \frac{1}{2} \log X + \frac{1}{2} \log q + \log \log X + O(1)$$

uniformly in $X \geq 3$ and q ; consequently $L(s, \chi)$ has no Landau–Siegel zero. This is a variant of the OpenAI proof, following Green and Kerr. The key arithmetic observation is that Frobenius at a prime inert in the quadratic field of χ is conjugation. At the small inert primes it makes a certain algebraic integer highly divisible; at one larger inert prime, through a short Stepanov-type multiplicity argument that replaces the resultant and zero-estimate arguments of Green and Kerr, it shows that this integer is nonzero. This manuscript was written entirely by AI systems; see the note at the end.

1. INTRODUCTION

Let χ be a primitive real Dirichlet character of conductor $q \geq 3$, and let $S_{\pm}(X)$ denote the sum of $(\log p)/p$ over the primes $p \leq X$ with $\chi(p) = \pm 1$. Implied constants are absolute.

Theorem 1. *For every $X \geq 3$, $S_{-}(X) \leq \frac{1}{2} \log X + \frac{1}{2} \log q + \log \log X + O(1)$.*

Corollary 2. *There is an absolute constant $c > 0$ such that every zero $\beta \in (0, 1)$ of $L(s, \chi)$ satisfies $(1 - \beta) \log q \geq c$.*

For fixed q one has $S_{-}(X) = \frac{1}{2} \log X + O_q(1)$, by Mertens’ theorem and $\sum_{p \leq X} \chi(p)(\log p)/p = O_q(1)$ (a consequence of $L(1, \chi) \neq 0$). So the coefficient $\frac{1}{2}$ of $\log X$ cannot be lowered, and the content of Theorem 1 is its uniformity in q ; a real zero near 1 would make $S_{-}(X)$ nearly $\log X$ for X a large power of q (§4).

The idea. Write $\chi = (d/\cdot)$ with d a fundamental discriminant, $|d| = q$, and for $\alpha = x + y\sqrt{d}$ put $\alpha' = x - y\sqrt{d}$. If $\chi(p) = -1$, conjugation on $\mathbb{Z}[\sqrt{d}]/p$ is the Frobenius, so at every point $(z, w) = (\alpha, \alpha')$

$$(1) \quad w \equiv z^p \pmod{p}, \quad \text{and} \quad \partial_z^{\kappa}(z^p - w) \equiv 0 \pmod{p} \quad (\kappa \geq 1).$$

Let δ be a nonzero minor of the matrix of z -derivatives of monomials $z^i w^j$ at the points (α, α') , α in an $N \times N$ box. At each small inert prime p , (1) makes δ divisible by about $p^{I/p}$, where I is the total z -degree. At one inert prime p_0 above all the z -degrees, (1) shows that such a δ exists: modulo p_0 the points lie on the graph $w = z^{p_0}$ of Frobenius, and a vanishing combination of rows would be a curve $F = 0$ meeting that graph with multiplicity at least t (the number of derivatives) at each of p_0 points, whereas $F(z, z^{p_0})$ has degree below $p_0 t$. A box of side $N \approx \sqrt{p_0}$ supplies the p_0 points. Ignoring the costs of the w -degrees and of differentiation, $|\delta|$ is then at most about H^I with $H \approx N\sqrt{q}$, and comparison gives $\sum_{p \text{ inert}} (\log p)/p \lesssim \log H \approx \frac{1}{2} \log p_0 + \frac{1}{2} \log q$. For a split prime Frobenius is the identity and $\partial_z(z^p - z) \equiv -1$ is a unit, so only S_{-} is controlled; that is the side a Landau–Siegel zero inflates.

Earlier work. The strategy—evaluating polynomials at Frobenius-twisted points of a box, and comparing the archimedean size of the resulting algebraic integer with its divisibility by inert

primes—is that of the OpenAI proof [6] (over $\mathbb{Q}(\sqrt{d}, \sqrt{2})$), as expounded by Banks [1]; we follow the variants of Kerr [4] and Green [3]. From Green we take the field $\mathbb{Q}(\sqrt{d})$, the grid of points (α, α') , the derivative evaluations, the height bound, the Leibniz-rule divisibility of the differentiated rows, and (as remarked in [3, §2]) the use of a nonsingular minor in place of a Gram determinant; from Kerr, the weighted monomials and the unitriangular change of basis to $z^v(z^p - w)^h$, which [3, Lemma 5.1] also uses in truncated, unweighted form. The grid has side about $\sqrt{p_0}$, as in [3] (where $N = 2X^{1/2}$).

The main change is the nonvanishing argument (Lemma 4): the multiplicity argument at the heart of Stepanov’s method [7], applied at an auxiliary inert prime. It plays the role of Masser’s zero estimate [5] in [4] and of the resultant argument of [3, Lemma 3.1, Appendix A]. Green’s lemma is more general: it applies to a much broader class of point configurations under geometric hypotheses, showing that the nonvanishing is a generic phenomenon and that the arithmetic lies in the divisibility. Lemma 4 instead exploits the arithmetic of these particular points; its merit is brevity, and that it reuses the fact (1) behind the divisibility. Further, because Kerr’s weighted monomials need no truncation in the change of basis, the weight ℓ may grow like $\log X$; this makes the costs of the w -degrees and of differentiation negligible and gives a bound uniform in X and q , a modest sharpening of the bound $(\frac{1}{2} + O(1/C)) \log X + O_C(1)$ for $X = q^C$ in [3, Theorem 1.1].

2. TWO LEMMAS

Let $\mathcal{O} = \mathbb{Z}[\sqrt{d}] = \mathbb{Z} \oplus \mathbb{Z}\sqrt{d}$.

Lemma 3. *Let p be an odd prime with $\chi(p) = -1$. Then $\mathcal{O}/p\mathcal{O}$ is a field with p^2 elements, and $\alpha^p \equiv \alpha' \pmod{p\mathcal{O}}$ for every $\alpha \in \mathcal{O}$.*

Proof. As d is not a square modulo p , $\mathcal{O}/p\mathcal{O} \cong \mathbb{F}_p[T]/(T^2 - d)$ is a field; and by Euler’s criterion $(x + y\sqrt{d})^p \equiv x + yd^{(p-1)/2}\sqrt{d} \equiv x - y\sqrt{d}$. $\square$

Lemma 4 (after Stepanov [7]). *Let k be a field of characteristic p , let $1 \leq t \leq p$, and let $F \in k[z, w]$ have $\deg_z F < p$ and $\deg_w F < t$. If $(\partial_z^\kappa F)(a, a^p) = 0$ for $0 \leq \kappa < t$ at p distinct points $a \in k$, then $F = 0$.*

Proof. Suppose $F \neq 0$ and put $G(z) = F(z, z^p)$. The monomial $z^i w^j$ becomes z^{i+pj} , and these exponents are distinct because $0 \leq i < p$; so $G \neq 0$ and $\deg G \leq (p-1) + p(t-1) = pt - 1$. As the derivative of z^p vanishes in characteristic p , the chain rule gives $G^{(\kappa)}(z) = (\partial_z^\kappa F)(z, z^p)$, so $G^{(\kappa)}(a) = 0$ for $\kappa < t$ at each of the p points. Since $\kappa! \neq 0$ in k for $\kappa < t \leq p$, Taylor expansion at a shows that $(z - a)^t$ divides G . Hence $\deg G \geq pt$, a contradiction. $\square$

Geometrically, in characteristic p the derivative ∂_z differentiates along the graph $w = z^p$ of Frobenius, and the proof counts the intersections of $F = 0$ with that graph.

3. PROOF OF THEOREM 1

Proposition 5. *Let $p_0 \geq 5$ be a prime with $\chi(p_0) = -1$, put $D = p_0 - 1$, and let $\ell \geq 2$ be an integer. Then*

$$S_-(D) \leq \left(\frac{1}{2} + \frac{7}{2\ell}\right) \log D + \frac{\ell+1}{2\ell} \log q + \log \ell + O(1).$$

Deduction of Theorem 1. If $X < q$, Mertens’ theorem gives $S_-(X) \leq \log X + O(1) \leq \frac{1}{2} \log X + \frac{1}{2} \log q + O(1)$. Let $X \geq q$, and let p_0 be the largest prime $\leq X$ with $\chi(p_0) = -1$; if there is none, or $p_0 \leq 3$, then $S_-(X) = O(1)$. Otherwise $S_-(X) = S_-(p_0 - 1) + (\log p_0)/p_0$, since by maximality the primes in $(p_0, X]$ contribute nothing. Apply Proposition 5 with $\ell = \lceil \log X \rceil \geq 2$: as $\log D \leq \log X$ and $\log q \leq \log X \leq \ell$, we have $\frac{7}{2\ell} \log D \leq \frac{7}{2}$, $\frac{\ell+1}{2\ell} \log q \leq \frac{1}{2} \log q + \frac{1}{2}$, and $\log \ell \leq \log \log X + \log 2$. $\square$

Proof of Proposition 5. Monomials. Let $s = \lfloor D/\ell \rfloor$, and let $\mathcal{M}$ be the set of $(i, j) \in \mathbb{Z}_{\geq 0}^2$ with $i + \ell j \leq D$; thus $i \leq D < p_0$ and $j \leq s$. Put $m = |\mathcal{M}|$, $I = \sum_{\mathcal{M}} i$ and $J = \sum_{\mathcal{M}} j$. As in [4] (where $\ell = 8$), trading z^p for w lowers the weight $i + \ell j$ when $p > \ell$, which keeps the change of basis below inside $\mathcal{M}$. Reflecting $i \mapsto D - \ell j - i$ for each fixed j gives $2I + \ell J = Dm$. Writing $i = \ell u + r$ with $0 \leq r < \ell$, for each r the pairs (u, j) fill a triangle $u + j \leq \lfloor (D - r)/\ell \rfloor$, symmetric in u and j ; so $I \geq \ell J$. Therefore

$$(2) \quad J \leq I/\ell, \quad m \leq 3I/D.$$

A nonzero algebraic integer. Let $N = \lceil \sqrt{p_0} \rceil$, and let M be the matrix over $\mathcal{O}$ with rows indexed by $\mathcal{M}$, ordered by increasing i , columns indexed by (x, y, κ) with $0 \leq x, y < N$ and $0 \leq \kappa \leq s$, and entries

$$M_{(i,j),(x,y,\kappa)} = \partial_z^\kappa (z^i w^j)(\alpha, \alpha') = (i)_\kappa \alpha^{i-\kappa} \alpha'^j \quad (\kappa \leq i), \quad \alpha = x + y\sqrt{d},$$

and 0 if $\kappa > i$; here $(i)_\kappa = i(i-1)\cdots(i-\kappa+1)$. Its rows are linearly independent modulo p_0 . Indeed, a relation among them over the field $\mathcal{O}/p_0\mathcal{O}$ is a polynomial $F = \sum_{\mathcal{M}} c_{ij} z^i w^j$ with $(\partial_z^\kappa F)(\bar{\alpha}, \bar{\alpha}^{p_0}) = 0$ for $\kappa \leq s$ (Lemma 3) at the $N^2 \geq p_0$ reductions $\bar{\alpha}$, which are distinct because $N < p_0$ and $1, \sqrt{d}$ are linearly independent over $\mathbb{F}_{p_0}$. As $\deg_z F \leq D < p_0$, $\deg_w F \leq s$ and $s+1 \leq p_0$, Lemma 4 with $t = s+1$ gives $F = 0$. Choose once and for all an $m \times m$ minor δ of M that is nonzero modulo p_0 . Then $\delta \neq 0$. Let δ' be its image under $\sqrt{d} \mapsto -\sqrt{d}$, which is the same minor with α and α' interchanged; then $n = |\delta\delta'| = |N_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}(\delta)|$ is a positive integer.

Its size. Put $H = 2N\sqrt{q}$ (as in [3, §4]). As $|\alpha|, |\alpha'| \leq N(1 + \sqrt{q}) \leq H$ and $(i)_\kappa \leq D^s$, every entry in row (i, j) of either matrix has modulus at most $D^s H^{i+j}$, and Hadamard's inequality gives

$$(3) \quad \log n \leq 2(I + J) \log H + 2ms \log D + m \log m.$$

Its divisibility. Let p be a prime with $\ell < p \leq D$ and $\chi(p) = -1$. For $(i, j) \in \mathcal{M}$ write $i = ph + v$ with $0 \leq v < p$, and replace the row of $z^i w^j$ by that of

$$z^v (z^p - w)^h w^j = z^i w^j + \sum_{\nu=1}^h (-1)^\nu \binom{h}{\nu} z^{i-p\nu} w^{j+\nu}$$

The monomials in the sum have smaller z -degree and smaller weight $i + \ell j - (p - \ell)\nu$, so they lie in $\mathcal{M}$. As the rows are ordered by increasing i , this adds to each row integer multiples of rows of smaller i , so it does not change δ , which uses every row. By Leibniz's rule, $\partial_z^\kappa (z^v (z^p - w)^h w^j)$ is an integer combination of products each containing h factors $\partial_z^\lambda (z^p - w)$ with $\lambda \geq 0$. Each such factor lies in $p\mathcal{O}$ at (α, α') : for $\lambda = 0$ by Lemma 3; for $1 \leq \lambda \leq p$ because $\partial_z^\lambda z^p = (p)_\lambda z^{p-\lambda}$ and $p \mid (p)_\lambda$; and for $\lambda > p$ it is zero. So the new row of (i, j) lies in $p^h \mathcal{O}$, and therefore $\delta \in p^{E_p} \mathcal{O}$; conjugating, $\delta' \in p^{E_p} \mathcal{O}$, so $p^{2E_p} \mid n$, where

$$(4) \quad E_p = \sum_{(i,j) \in \mathcal{M}} \left\lfloor \frac{i}{p} \right\rfloor \geq \frac{I}{p} - m.$$

Comparison. The coprime p^{2E_p} all divide n , so by (4) and Chebyshev's bound $\sum_{p \leq D} \log p \ll D$,

$$2I \sum_{\substack{\ell < p \leq D \\ \chi(p) = -1}} \frac{\log p}{p} \leq 2 \sum_p E_p \log p + O(mD) \leq \log n + O(mD).$$

Insert (3) and divide by $2I$. By (2) and $m \leq (D+1)^2$, we have $(I+J)/I \leq 1 + \frac{1}{\ell}$, $ms/I \leq \frac{3}{\ell}$, and $(m \log m)/I$, mD/I are $O(1)$. The primes $p \leq \ell$ add at most $\log \ell + O(1)$ to $S_-(D)$, by Mertens' theorem; so

$$S_-(D) \leq \log H + \frac{\log H + 3 \log D}{\ell} + \log \ell + O(1).$$

Here $\log H$ is the height of the grid, the terms with $1/\ell$ are the costs of the w -degrees and of differentiation, and $\log \ell$ accounts for the small primes set aside. Since $N \leq 2\sqrt{D}$, we have $\log H \leq \frac{1}{2} \log(Dq) + \log 4$, and the proposition follows. $\square$

4. PROOF OF COROLLARY 2

This deduction is standard (cf. [4, Lemma 1]); we include it for completeness. Let $L(\beta, \chi) = 0$ with $0 < \beta < 1$, let $X \geq 3$, and put $\sigma = 1 + 1/\log X \in (1, 2]$. Taking real parts in the formula for L'/L that comes from the Hadamard product [2, p. 89],

$$-\frac{L'}{L}(\sigma, \chi) = \frac{1}{2} \log \frac{q}{\pi} + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma + \mathfrak{a}}{2} \right) - \sum_{\rho} \operatorname{Re} \frac{1}{\sigma - \rho}, \quad \chi(-1) = (-1)^{\mathfrak{a}},$$

where ρ runs over the nontrivial zeros. The gamma term is bounded, and every term of the sum is positive because $\operatorname{Re} \rho \leq 1 < \sigma$. Keeping only $\rho = \beta$, adding $-\frac{\zeta'}{\zeta}(\sigma) \leq \frac{1}{\sigma-1} + O(1)$, and using $\frac{1}{\sigma-1} - \frac{1}{\sigma-\beta} \leq \frac{1-\beta}{(\sigma-1)^2} = (1-\beta)(\log X)^2$,

$$\sum_{n \geq 1} \frac{\Lambda(n)(1 + \chi(n))}{n^{\sigma}} \leq \frac{1}{2} \log q + (1-\beta)(\log X)^2 + O(1).$$

All terms on the left are nonnegative, and a prime $p \leq X$ with $\chi(p) = 1$ contributes $2(\log p)p^{-\sigma} \geq 2e^{-1}(\log p)/p$. Hence $S_+(X) \leq \frac{e}{2}(1-\beta)(\log X)^2 + O(\log q)$. On the other hand, the primes with $\chi(p) = 0$ divide q , so Mertens' theorem gives $S_+(X) + S_-(X) \geq \log X - \log q - O(1)$. Combining these with Theorem 1 and $\log \log X \leq \frac{1}{4} \log X + O(1)$,

$$\frac{1}{4} \log X \leq C \log q + \frac{e}{2}(1-\beta)(\log X)^2$$

for an absolute constant C . Taking $X = q^{\lambda}$ with $\lambda = \max(1, 8C)$ gives $(1-\beta) \log q \geq 1/(4e\lambda)$. $\square$

Authorship. This manuscript is AI-generated. Everything in it that is not taken from the cited works—in particular the use of an auxiliary inert prime and the arrangement of the proof—was found and written by AI systems (GPT-6 Pro and Claude Opus 5.5), building on [6, 1, 3, 4]. The human who prepared the manuscript contributed no mathematical arguments, and only passed material between the systems and asked for revisions. The proofs have been checked by several independent AI reviews, not by a human. The manuscript has not been refereed.

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