

LANDAU–SIEGEL ZEROS AND A STEPANOV-TYPE NONVANISHING LEMMA

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ABSTRACT. Let χ be a primitive real character of conductor $q \geq 3$. We prove Barreto’s bound

$$\sum_{\substack{p \leq X \\ \chi(p) = -1}} \frac{\log p}{p} \leq \frac{1}{2} \log X + \frac{1}{4} \log q + \log \log X + O(1)$$

uniformly in $X \geq 3$ and q , with an effective implied constant, and hence that every zero $\beta \in (0, 1)$ of $L(s, \chi)$ satisfies $(1 - \beta) \log q \geq 1 - O(\log \log q / \log q)$, so there is no Landau–Siegel zero. As in OpenAI’s proof, the norm of an interpolation determinant is bounded above by the size of its entries and below by its divisibility by high powers of the odd inert primes. Here the points are conjugate pairs (α, α') of quadratic integers in Barreto’s rectangular box, which gives the $\frac{1}{4}$. In our proof a single congruence, that conjugation is the Frobenius modulo an odd inert prime, gives both the divisibility and the nonvanishing. Modulo a large inert prime p_0 every point lies on the graph $w = z^{p_0}$, and as z^{p_0} has derivative zero modulo p_0 , a linear relation among the rows of the interpolation matrix would give a polynomial in z with more roots, counted with multiplicity, than its degree, as in Stepanov’s method.

1. INTRODUCTION

A Landau–Siegel zero is a real zero β of a quadratic Dirichlet L -function $L(s, \chi)$ with $(1 - \beta) \log q$ small, q being the conductor. Other Dirichlet L -functions have none [4, Ch. 14]. Excluding such zeros gives the effective bound $L(1, \chi) \gg 1 / \log q$ in place of Siegel’s ineffective $L(1, \chi) \gg_{\epsilon} q^{-\epsilon}$, and so, when χ is odd, an effective bound $\gg \sqrt{q} / \log q$ for the class number of $\mathbb{Q}(\sqrt{-q})$. Such zeros were recently excluded by OpenAI [11], as expounded by Banks [1], and since then by Kerr [8], Green [7], Barreto [2] and van Dijk [14]. All of these proofs rest on divisibility by high powers of the inert primes, and they differ in their points, weights and nonvanishing arguments. This paper proves nonvanishing with the congruence that gives the divisibility, by a Stepanov-type root count at an auxiliary inert prime (Lemma 4). Version 1 of this paper [6] introduced this root count and a weight growing with X , and with a square box of points proved the bound of Theorem 1 below with $\frac{1}{2} \log q$ in place of $\frac{1}{4} \log q$. The present version takes Barreto’s rectangular box, and with it proves his bounds.

Let χ be a primitive real Dirichlet character of conductor $q \geq 3$, and let $S_{\pm}(X)$ denote the sum of $(\log p)/p$ over the primes $p \leq X$ with $\chi(p) = \pm 1$. Implied constants are absolute and effective, but we do not compute them. We use Mertens’ theorem $\sum_{p \leq x} (\log p)/p = \log x + O(1)$ and Chebyshev’s bound $\sum_{p \leq x} \log p \ll x$.

Theorem 1 (Barreto [2]). *For $X \geq 3$ we have*

$$S_{-}(X) \leq \frac{1}{2} \log X + \frac{1}{4} \log q + \log \log X + O(1).$$

Version 3, 22 October 2026. All versions and corrections:
<https://jacobpescador.github.io/ai-math/papers/landau-siegel-stepanov/>.

Corollary 2 (Barreto [2]). *Every zero $\beta \in (0, 1)$ of $L(s, \chi)$ satisfies*

$$(1 - \beta) \log q \geq 1 - O\left(\frac{\log \log q}{\log q}\right) \quad \text{and} \quad (1 - \beta) \log q \gg 1.$$

Barreto's post is no longer available, and Theorem 1 and Corollary 2 are his results, stated in our notation.

A real zero close to 1 would make $\chi(p) = -1$ for most primes in a long range, and so make $S_-(X)$ nearly $\log X$, contrary to Theorem 1. For fixed q , Mertens' theorem and $L(1, \chi) \neq 0$ give $S_-(X) = \frac{1}{2} \log X + O_q(1)$, so the $\frac{1}{2}$ cannot be lowered, and the content of Theorem 1 is its uniformity in q .

The idea. Write χ as the Kronecker symbol $(d/\cdot)$ with d a fundamental discriminant, $|d| = q$, and for $\alpha = x + y\sqrt{d}$ put $\alpha' = x - y\sqrt{d}$. If p is odd and $\chi(p) = -1$, so that p is inert in $\mathbb{Q}(\sqrt{d})$, then conjugation on $\mathbb{Z}[\sqrt{d}]/p$ is the Frobenius, so at every point $(z, w) = (\alpha, \alpha')$

$$(1) \quad w \equiv z^p \pmod{p}, \quad \text{and} \quad \partial_z^\kappa(z^p - w) \equiv 0 \pmod{p} \quad (\kappa \geq 1).$$

The plan is to build a determinant δ with entries of controlled size that is a nonzero algebraic integer divisible by high powers of many inert primes. A nonzero integer has absolute value at least the product of the prime powers dividing it, so the divisibility bounds the norm of δ from below, Hadamard's inequality bounds it from above, and comparing the two gives an upper bound for S_- .

Fix an inert prime p_0 and a weight ℓ . The matrix has a row for each monomial $z^i w^j$ with $i + \ell j \leq p_0 - 1$. The weight keeps the w -degrees small, as they add to the size of the entries but not to the divisibility, which comes from the z -degrees. The matrix has a column for each point (α, α') with α in a box $0 \leq x < N_1$, $0 \leq y < N_2$ whose sides are chosen below, and each order of z -derivative up to the largest w -degree, about p_0/ℓ , as the root count below requires. Each entry is that derivative of the monomial at the point. Take δ to be a minor that uses every row, and δ' its conjugate, so that $\delta\delta'$ is an integer. There are about $p_0^2/(2\ell)$ rows but only about p_0 points in the box, and the derivatives supply the missing columns without enlarging the box of points.

Divisibility. At an inert prime p with $\ell < p < p_0$, the change of basis $z^i w^j \mapsto z^v (z^p - w)^h w^j$, where $i = ph + v$ with $0 \leq v < p$, trades z -degree for w -degree and leaves δ unchanged. By (1) the new row is divisible by p^h , so δ is divisible by about $p^{I/p}$, where $I = \sum i$.

Nonvanishing. At p_0 every z -degree is below p_0 , so there is nothing to rewrite, and (1) is used a second time. Modulo p_0 the points lie on the graph $w = z^{p_0}$ of Frobenius, and since $\partial_z(z^{p_0} - w) \equiv 0$, differentiating in z commutes with substituting $w = z^{p_0}$. If the box holds p_0 points distinct modulo p_0 , a relation among the rows modulo p_0 therefore becomes a polynomial in z with more roots, counted with multiplicity, than its degree (Lemma 4), so δ can be chosen nonzero. The derivatives thus serve both steps: they keep the divisibility at the small inert primes, and at p_0 they supply the multiplicities. This argument uses characteristic p_0 . Over $\mathbb{C}$, the substitution $w = g(z)$ commutes with ∂_z only for constant g , and each line $w = c$ contains at most one of the points. Modulo p_0 the choice $g = z^{p_0}$ is also available, and its graph contains all of them.

Size. Call $\max(|\alpha|, |\alpha'|)$ the height of the point (α, α') . It controls the entries of δ and δ' , and the root count requires p_0 points, so we choose the box to minimise their height. A box with sides N_1, N_2 has $N_1 N_2$ points, each of height at most $N_1 + N_2 \sqrt{q}$. The best shape balances the two terms: $N_1 \approx p_0^{1/2} q^{1/4}$ and $N_2 \approx p_0^{1/2} q^{-1/4}$, with height $H \approx p_0^{1/2} q^{1/4}$. The points stay distinct modulo p_0 when $q \leq p_0^2$. In that range, up to constants, no set of p_0 points of $\mathbb{Z}[\sqrt{d}]$ does better, since $\#\{\alpha : \max(|\alpha|, |\alpha'|) \leq H\} \ll (1 + H)(1 + H/\sqrt{q})$.

Heuristically, Hadamard's inequality gives $\log|\delta\delta'| \lesssim 2I \log H$, and the divisibility gives $\log|\delta\delta'| \gtrsim 2I \sum (\log p)/p$ over the inert $p \in (\ell, p_0)$, so

$$\sum_{\substack{\ell < p < p_0 \\ \chi(p) = -1}} \frac{\log p}{p} \lesssim \log H \approx \frac{1}{2} \log p_0 + \frac{1}{4} \log q.$$

The right side is at most $\log p_0$, the bound given by Mertens' theorem, exactly when $q \leq p_0^2$. Taking p_0 to be the largest inert prime up to X , Mertens' theorem alone gives Theorem 1 for larger q .

In the proof, the w -degrees and the derivatives add $O(\log p_0/\ell)$ to this bound, and the primes $p \leq \ell$, which the proof sets aside, add $\log \ell$. We take $\ell \approx \log p_0$. Without derivatives, about $p_0^2/(2\ell)$ points would be needed, of height about $p_0 q^{1/4}/\sqrt{\ell}$. Adding $\log \ell$, this size estimate would give only $S_- \lesssim \log p_0 + \frac{1}{4} \log q + \frac{1}{2} \log \ell$, worse than Mertens' bound. The $\frac{1}{4}$ gives the constant 1 in Corollary 2 (Remark 7).

Why inert primes. The comparison cannot count the split primes as well, for it would then bound $\sum_{\ell < p < p_0} (\log p)/p \approx \log p_0$ by $\frac{1}{2} \log p_0 + \frac{1}{4} \log q$, which is smaller once q is well below p_0^2 . Something in the construction must exclude them, and it is the derivatives. At an odd split prime, $\mathbb{Z}[\sqrt{d}]/p \cong \mathbb{F}_p \times \mathbb{F}_p$ with conjugation swapping the two factors, so no polynomial g gives $\alpha' \equiv g(\alpha) \pmod{p}$ for all $\alpha \in \mathbb{Z}[\sqrt{d}]$. What holds there is $z^p \equiv z$ and $w^p \equiv w$. With values alone the first would serve a split prime as $z^p \equiv w$ serves an inert one: replacing z^i by $z^v(z^p - z)^h$, with $i = ph + v$, makes the row divisible by p^h . Derivatives undo this. Modulo p we have $\partial_z(z^p - w) \equiv 0$, so a differentiated factor $z^p - w$ stays divisible by p , while $\partial_z(z^p - z) \equiv -1$ is a unit, so each order of derivative can use up one factor $z^p - z$. For $p > \ell$ a row has $h = \lfloor i/p \rfloor$ of these factors, at most the largest order of derivative, about p_0/ℓ , so counting them guarantees nothing at p . The relation $w^p \equiv w$ survives differentiation, but it can only spend w -degree, p units for each factor of p , and the weight keeps the w -degrees to about $1/\ell$ of the z -degrees, so it adds at most about $(\log p_0)/\ell$. The derivatives that save the $\frac{1}{2}$ thus keep the split primes above ℓ out of the count, and the argument bounds S_- , the sum that a Landau-Siegel zero makes large.

2. TWO LEMMAS

Let $\mathcal{O} = \mathbb{Z}[\sqrt{d}] = \mathbb{Z} \oplus \mathbb{Z}\sqrt{d}$.

Lemma 3. *Let p be an odd prime with $\chi(p) = -1$. Then $\mathcal{O}/p\mathcal{O}$ is a field, and $\alpha^p \equiv \alpha' \pmod{p\mathcal{O}}$ for every $\alpha \in \mathcal{O}$.*

Proof. As $(d/p) = \chi(p) = -1$, d is not a square modulo p , so $\mathcal{O}/p\mathcal{O} \cong \mathbb{F}_p[T]/(T^2 - d)$ is a field. By the binomial theorem, Fermat's little theorem and Euler's criterion, $(x + y\sqrt{d})^p \equiv x + yd^{(p-1)/2}\sqrt{d} \equiv x - y\sqrt{d} \pmod{p\mathcal{O}}$. (Oddness is needed: $\mathcal{O}$ need not be maximal at 2.) $\square$

Lemma 4 (after Stepanov [12]). *Let k be a field of characteristic p , let $1 \leq t \leq p$, and let $F \in k[z, w]$ have $\deg_z F < p$ and $\deg_w F < t$. If $(\partial_z^\kappa F)(a, a^p) = 0$ for $\kappa < t$ at p distinct $a \in k$, then $F = 0$.*

Proof. Suppose $F \neq 0$ and put $G(z) = F(z, z^p)$. The monomial $z^i w^j$ becomes z^{i+pj} , and these exponents are distinct because $0 \leq i < p$, so $G \neq 0$ and $\deg G \leq (p-1) + p(t-1) = pt - 1$. As the derivative of z^p vanishes in characteristic p , the chain rule gives $G^{(\kappa)}(z) = (\partial_z^\kappa F)(z, z^p)$, so $G^{(\kappa)}(a) = 0$ for $\kappa < t$ at each of the p points. Writing $G = \sum_\kappa c_\kappa (z-a)^\kappa$ gives $G^{(\kappa)}(a) = \kappa! c_\kappa$, and $\kappa! \neq 0$ in k for $\kappa < t \leq p$, so $(z-a)^t$ divides G for each of the p points. The $(z-a)^t$ are pairwise coprime, contradicting $\deg G < pt$. $\square$

Lemma 4 is the multiplicity count of Stepanov's method. Stepanov constructs a nonzero auxiliary polynomial vanishing to high order at a set of points and uses the count to bound the size of the set. Here the points are given, and the count shows that $F = 0$. The count is exact on the rectangle $i < p$, $j < t$: the exponents $i + pj$ fill $0, \dots, pt - 1$ once each, so by Hermite interpolation the pt values $(\partial_z^\kappa F)(a, a^p)$ can be arbitrary. In §3 the lemma is applied with $p = p_0$ to the triangle $i + \ell j \leq D$ instead, which, unlike the rectangle, is preserved by the change of basis at each inert prime in $(\ell, D]$.

3. PROOF OF THEOREM 1

Proposition 5. *Let $p_0 \geq 5$ be a prime with $\chi(p_0) = -1$ and $q \leq p_0^2$, put $D = p_0 - 1$, and let $\ell \geq 2$ be an integer. Then*

$$S_-(D) \leq \left(\frac{1}{2} + \frac{7}{2\ell}\right) \log D + \frac{\ell+1}{4\ell} \log q + \log \ell + O(1).$$

Deduction of Theorem 1. Let p_0 be the largest prime $\leq X$ with $\chi(p_0) = -1$. If there is no such prime, or $p_0 \leq 3$, then $S_-(X) = O(1)$. Otherwise $S_-(X) = S_-(p_0)$ by maximality. If $q > p_0^2$, then Mertens' theorem gives $S_-(X) \leq \log p_0 + O(1) \leq \frac{1}{2} \log X + \frac{1}{4} \log q + O(1)$. If $q \leq p_0^2$, then $S_-(X) = S_-(D) + O(1)$, and we apply Proposition 5 with $\ell = \lceil \log p_0 \rceil \geq 2$. As $\log D \leq \ell$ and $\log q \leq 2\ell$, the terms $7 \log D/(2\ell)$ and $\log q/(4\ell)$ are $O(1)$, and $\log \ell \leq \log \log X + 1$. $\square$

Proof of Proposition 5. Monomials. Let $s = \lfloor D/\ell \rfloor$, and let $\mathcal{M}$ be the set of $(i, j) \in \mathbb{Z}_{\geq 0}^2$ with $i + \ell j \leq D$, so that $i \leq D < p_0$ and $j \leq s$. Put $m = |\mathcal{M}|$, $I = \sum_{\mathcal{M}} i$ and $J = \sum_{\mathcal{M}} j$. Reflecting $i \mapsto D - \ell j - i$ for each fixed j gives $2I + \ell J = Dm$. Writing $i = \ell u + r$ with $0 \leq r < \ell$, for each r the pairs (u, j) fill a triangle $u + j \leq \lfloor (D - r)/\ell \rfloor$, symmetric in u and j , so $\sum u = \sum j$ and $I = \sum (\ell u + r) \geq \ell \sum u = \ell J$. Therefore, as $Dm = 2I + \ell J \leq 3I$,

$$(2) \quad J \leq I/\ell, \quad m \leq 3I/D.$$

A nonzero algebraic integer. Let $N_1 = \lceil p_0^{1/2} q^{1/4} \rceil$ and $N_2 = \lceil p_0^{1/2} q^{-1/4} \rceil$. Then $N_1 N_2 \geq p_0$. Moreover, $N_1 \leq p_0$, because $p_0^{1/2} q^{1/4} \leq p_0$ when $q \leq p_0^2$, and $N_2 \leq \lceil p_0^{1/2} \rceil \leq p_0$. Let A be the matrix over $\mathcal{O}$ with rows indexed by $\mathcal{M}$, ordered by increasing i , columns indexed by (x, y, κ) with $0 \leq x < N_1$, $0 \leq y < N_2$ and $0 \leq \kappa \leq s$, and entries

$$A_{(i,j),(x,y,\kappa)} = \partial_z^\kappa (z^i w^j)(\alpha, \alpha') = (i)_\kappa \alpha^{i-\kappa} \alpha'^j \quad (\kappa \leq i), \quad \alpha = x + y\sqrt{d},$$

and 0 if $\kappa > i$, where $(i)_\kappa = i(i-1)\cdots(i-\kappa+1)$. Its rows are linearly independent modulo p_0 . Indeed, a relation among them over the field $\mathcal{O}/p_0\mathcal{O}$ is a polynomial $F = \sum_{\mathcal{M}} c_{ij} z^i w^j$ with $(\partial_z^\kappa F)(\bar{\alpha}, \bar{\alpha}^{p_0}) = 0$ for $\kappa \leq s$ (Lemma 3) at the $N_1 N_2 \geq p_0$ reductions $\bar{\alpha}$, which are distinct because $N_1, N_2 \leq p_0$ and $1, \sqrt{d}$ are linearly independent over $\mathbb{F}_{p_0}$. As $\deg_z F \leq D < p_0$, $\deg_w F \leq s$ and $s+1 \leq p_0$, Lemma 4 with $t = s+1$ gives $F = 0$. Hence we may fix an $m \times m$ minor $\delta \notin p_0\mathcal{O}$ of A , which does not depend on the primes p below. Let δ' be its image under $\sqrt{d} \mapsto -\sqrt{d}$, which is the same minor with α and α' interchanged. Then $n = |\delta\delta'|$ is a positive integer, as $\delta\delta' \in \mathbb{Z} \setminus \{0\}$. (We pass to the norm because $|\delta|$ itself can be small when $d > 0$.)

Its size. Put $H = N_1 + N_2\sqrt{q}$. As $|\alpha|, |\alpha'| \leq x + y\sqrt{q} < H$ and $(i)_\kappa \leq D^s$, every entry in row (i, j) of the matrices of δ and δ' has modulus at most $D^s H^{i+j}$, so that row has length at most $\sqrt{m} D^s H^{i+j}$. Hadamard's inequality, which bounds a determinant by the product of its row lengths, gives

$$(3) \quad \log n \leq 2(I + J) \log H + 2ms \log D + m \log m.$$

Its divisibility. Let p be a prime with $\ell < p \leq D$ and $\chi(p) = -1$. As $\ell \geq 2$, p is odd, so Lemma 3 applies. For $(i, j) \in \mathcal{M}$ write $i = ph + v$ with $0 \leq v < p$, and replace the row of $z^i w^j$ by that of

$$z^v (z^p - w)^h w^j = z^i w^j + \sum_{\nu=1}^h (-1)^\nu \binom{h}{\nu} z^{i-p\nu} w^{j+\nu}.$$

The monomials in the sum have smaller z -degree and smaller $i + \ell j$, namely $i + \ell j - (p - \ell)\nu$, so they lie in $\mathcal{M}$. Doing this for every row adds to each row integer multiples of rows of smaller i , a unitriangular change, so δ , which uses every row, is unchanged. By Leibniz's rule, $\partial_z^\lambda (z^v (z^p - w)^h w^j)$ is an integer combination of products each containing h factors $\partial_z^\lambda (z^p - w)$ with $\lambda \geq 0$. Each such factor lies in $p\mathcal{O}$ at (α, α') . For $\lambda = 0$ this is Lemma 3. For $1 \leq \lambda \leq p$ it holds because

$\partial_z^\lambda z^p = (p)_\lambda z^{p-\lambda}$ and $p \mid (p)_\lambda$, and for $\lambda > p$ the factor is zero. So the new row of (i, j) lies in $p^h \mathcal{O}$, hence $\delta \in p^{E_p} \mathcal{O}$. Conjugating, $\delta' \in p^{E_p} \mathcal{O}$, so $p^{2E_p} \mid n$ with

$$(4) \quad E_p = \sum_{(i,j) \in \mathcal{M}} \left\lfloor \frac{i}{p} \right\rfloor \geq \frac{I}{p} - m.$$

Comparison. As the pairwise coprime p^{2E_p} all divide n , multiplying (4) by $2 \log p$, summing over p and using Chebyshev's bound $\sum_{p \leq D} \log p \ll D$ give

$$2I \sum_{\substack{\ell < p \leq D \\ \chi(p) = -1}} \frac{\log p}{p} \leq 2 \sum_p E_p \log p + O(mD) \leq \log n + O(mD).$$

Insert (3) and divide by $2I$. By (2), we have $(I+J)/I \leq 1 + 1/\ell$, $ms/I \leq 3/\ell$ and $mD/I \leq 3$, and also $(m \log m)/I \leq 3(\log m)/D = O(1)$ since $m \leq (D+1)^2$. Mertens' theorem gives $\log \ell + O(1)$ for the primes $p \leq \ell$, so

$$(5) \quad S_-(D) \leq \log H + \frac{\log H + 3 \log D}{\ell} + \log \ell + O(1).$$

Finally, $N_1 \leq p_0^{1/2} q^{1/4} + 1$ and $N_2 \sqrt{q} \leq p_0^{1/2} q^{1/4} + \sqrt{q}$, and $1 \leq \sqrt{q} \leq p_0^{1/2} q^{1/4}$ as $q \leq p_0^2$. Hence $H \leq 4p_0^{1/2} q^{1/4}$ and $\log H \leq \frac{1}{2} \log D + \frac{1}{4} \log q + O(1)$, and inserting this in (5) gives

$$S_-(D) \leq \left(1 + \frac{1}{\ell}\right) \left(\frac{1}{2} \log D + \frac{1}{4} \log q\right) + \frac{3}{\ell} \log D + \log \ell + O(1),$$

which is the proposition, as $\frac{1}{2} + \frac{1}{2\ell} + \frac{3}{\ell} = \frac{1}{2} + \frac{7}{2\ell}$. $\square$

Remark 6. For a suitable family of fields, no choice of the monomials or of the primes used removes the $\log \log X$ from an argument that uses (3) and the divisibility that Lemma 3 gives along the lines $i + pj = c$. Let $\mathcal{M}$ be any finite set of exponents with $i \leq D$ and $I > 0$, and p an odd inert prime. Expanding $w^j = (z^p + (w - z^p))^j$ as in the proof, the t_c points of $\mathcal{M}$ on a line $i + pj = c$ give $\delta \in p^{E_p} \mathcal{O}$ with $E_p = \sum_c \binom{t_c}{2}$, which is (4) for the triangle when $p > \ell$. These points have distinct $j \geq 0$, summing to at least $\binom{t_c}{2}$, and distinct $i = c - pj \geq 0$, congruent modulo p , summing to at least $p \binom{t_c}{2}$. Hence $E_p \leq \min(J, I/p)$.

Now let $L = \frac{1}{4} \log D$. By the Chinese remainder theorem there is an $a \equiv 1 \pmod{8}$ with $a \leq 8p_0 \prod_{3 \leq p \leq L} p = p_0^{5/4+o(1)}$ that is a quadratic nonresidue modulo p_0 and modulo every odd prime up to L . None of these primes divides a , so the squarefree part d of a has the same residue symbols, $d \equiv a \equiv 1 \pmod{8}$, and d is a fundamental discriminant with $q = d \leq p_0^2$ once p_0 is large, for which p_0 and every odd prime up to L are inert. With $\rho = J/I$, the comparison in the proof bounds $S_-(D)$ by $\log H$ plus at least $\rho \log H + \sum (1/p - E_p/I) \log p$ over the inert $p \leq D$, where $E_p = 0$ at a prime not used. As $E_p/I \leq \min(\rho, 1/p)$ and $\log H \geq \frac{1}{2} \log D$, the inert primes up to $x = \min(L, 1/\rho)$ ($x = L$ if $J = 0$) make this at least $\frac{\rho}{2} \log D + \log x - O(1)$, which is $\log \log D - O(1)$ or more whether $\rho \leq 1/L$ or not. The conductors of this family grow with D ; the remark says nothing about one fixed field. Removing the $\log \log X$ needs another mechanism, and would not improve Corollary 2, whose proof loses $O(\log \log q)$ at the primes dividing q in any case.

4. PROOF OF COROLLARY 2

A zero β near 1 makes $-L'/L(\sigma, \chi)$ very negative just to the right of 1, while Theorem 1 bounds it below.

Proof of Corollary 2. Lower bound for $-L'/L$. Let $T(x) = \sum_{p \leq x} \chi(p)(\log p)/p = S_+(x) - S_-(x)$, so that $T(x) = 0$ for $1 \leq x < 2$. The primes with $\chi(p) = 0$ divide q , and

$$\sum_{p|q} \frac{\log p}{p} \leq \sum_{p \leq \log q} \frac{\log p}{p} + \frac{1}{\log q} \sum_{p|q} \log p \leq \log \log q + O(1).$$

So Mertens' theorem gives $T(x) \geq \log x - 2S_-(x) - O(\log \log q)$. Writing $\log^+ = \max(\log, 0)$, Theorem 1 then gives some $K = \frac{1}{2} \log q + O(\log \log q)$ with $K \geq \frac{1}{2} \log q$ and

$$(6) \quad T(x) \geq -K - 2 \log^+ \log x \quad (x \geq 1),$$

which is trivial for $x < 3$, where $T(x) \geq -\frac{1}{2} \log 2 > -K$. Let $0 < u \leq 1$ and $\sigma = 1 + u$. The logarithmic derivative is an average of T over logarithmic scales. By partial summation, and since the prime powers contribute $O(1)$ (as $\sum_p (\log p)/(p(p-1))$ converges),

$$-\frac{L'}{L}(\sigma, \chi) = u \int_1^\infty T(x) x^{-u-1} dx + O(1).$$

The weight $u x^{-u-1}$ has total mass 1 and satisfies $u^2 \int_1^\infty (\log x) x^{-u-1} dx = 1$. Since $\log y \leq y$, we have $\log^+ \log x \leq \log(1/u) + u \log x$, and so (6) gives

$$(7) \quad -\frac{L'}{L}(\sigma, \chi) \geq -\frac{1}{2} \log q - 2 \log \frac{1}{u} - O(\log \log q).$$

Upper bound for $-L'/L$. Let $L(\beta, \chi) = 0$ with $0 < \beta < 1$. Taking real parts in the formula for L'/L that comes from the Hadamard product [4, Ch. 12, (17)–(18), and Ch. 14, p. 89, (2)],

$$-\frac{L'}{L}(\sigma, \chi) = \frac{1}{2} \log \frac{q}{\pi} + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma + \mathfrak{a}}{2} \right) - \sum_{\rho} \operatorname{Re} \frac{1}{\sigma - \rho}, \quad \chi(-1) = (-1)^{\mathfrak{a}}, \quad \mathfrak{a} \in \{0, 1\},$$

where ρ runs over the nontrivial zeros of $L(s, \chi)$, counted with multiplicity. The gamma term is bounded, as $(\sigma + \mathfrak{a})/2 \in (\frac{1}{2}, \frac{3}{2}]$, and every term of the sum is positive because $\operatorname{Re} \rho \leq 1 < \sigma$. Keeping only $\rho = \beta$ and using (7),

$$(8) \quad \frac{1}{u + 1 - \beta} \leq \log q + 2 \log \frac{1}{u} + O(\log \log q) \quad (0 < u \leq 1).$$

Conclusion. Put $\mathcal{Q} = \log q$ and $u = \mathcal{Q}^{-2}$, so that $u\mathcal{Q} = 1/\mathcal{Q}$ and $\log(1/u) = 2 \log \mathcal{Q}$. By (8),

$$(1 - \beta)\mathcal{Q} \geq \frac{\mathcal{Q}}{\mathcal{Q} + O(\log \mathcal{Q})} - \frac{1}{\mathcal{Q}} \geq 1 - O\left(\frac{\log \mathcal{Q}}{\mathcal{Q}}\right).$$

This bound is empty for small q . To get $(1 - \beta) \log q \gg 1$ for every q , take $u = \eta/\mathcal{Q}$ instead, with $0 < \eta \leq 1$. Since $\log(1/u) = \log(1/\eta) + \log \mathcal{Q} \leq \log(1/\eta) + \mathcal{Q}$, (8) gives $1/(u + 1 - \beta) \ll (1 + \log(1/\eta))\mathcal{Q}$. Hence $(1 - \beta)\mathcal{Q} \geq c_0/(1 + \log(1/\eta)) - \eta$ with $c_0 > 0$ absolute, and this is positive once η is small enough. $\square$

Remark 7. The constant 1 in Corollary 2 is $1/(\frac{1}{2} + 2 \cdot \frac{1}{4})$: the $\frac{1}{2}$ comes from the conductor in the Hadamard formula, and $2 \cdot \frac{1}{4}$ from Theorem 1, as S_- appears twice in $T = (S_+ + S_-) - 2S_-$. The constant also depends on the coefficient of $\log X$ in Theorem 1 being exactly $\frac{1}{2}$, so that $2S_-$ cancels the $\log x$ of Mertens' theorem. With $\frac{1}{2} + \varepsilon$ in its place, the right side of (8) would gain a term $2\varepsilon/u$, and the argument would give a constant below 1.

5. RELATION TO OTHER PROOFS

The OpenAI proof [11], expounded by Banks [1], works over $\mathbb{Q}(\sqrt{d_0}, \sqrt{2})$, where d_0 is the square-free part of d . As both note, it is an interpolation-determinant argument [9], with the smallness p -adic rather than archimedean, as in Dobrowolski's work on Lehmer's problem [5] (a connection noted by K. Pratt in the comments to [13]), whose determinant form is due to Cantor and Straus [3].

Divisibility. All the proofs use the congruence $w \equiv z^p \pmod{p}$. Green [7] works in $\mathbb{Q}(\sqrt{d})$ with the points (α, α') , takes z -derivatives and applies Leibniz’s rule, and orders the rows by z -degree. We follow him in this. Where his main proof uses a Gram determinant, we use a nonzero minor, as a remark in [7, version 2, §2] also suggests. The change of basis to $z^v(z^p - w)^h$ comes from [11], in Kerr’s form [8].

Points and weights. A square box of points, as in Green’s proof, gives $\frac{1}{2} \log q$, and Barreto’s [2] rectangular box gives $\frac{1}{4} \log q$. A fixed weight, as in OpenAI’s and Kerr’s proofs, gives the bound at X a fixed power of q . The weight growing with X of version 1 gives it uniformly in X , at the cost of $\log \log X$ (Remark 6).

Nonvanishing. OpenAI [11] uses an interpolation lemma with a greedy choice of monomial rows. Kerr [8] constructs an auxiliary polynomial by the pigeonhole principle in place of a determinant, and Masser’s zero estimate [10] shows that it cannot vanish at all the points. Green [7, Lemma 3.1] bounds multiplicities by resultants at any points with distinct second coordinates satisfying an incidence bound, which Schwartz–Zippel supplies. Barreto [2] uses a Wronskian and a minor in $\mathbb{Z}$. Van Dijk [14] proves nonvanishing by the same root count, and his version 2 makes his proof explicit for class number one: an imaginary quadratic field of class number one has discriminant of absolute value below $8 \cdot 10^9$, and a finite sieve then leaves exactly the nine known fields.

AUTHORSHIP

The mathematics and text were produced by the AI systems named above, building on their version 1 [6] and the work cited above. Jacob Pescador initiated and coordinated the project and maintains this posting.

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